

Question

A spin-half particle is in a linear superposition $0.5|\uparrow\rangle + \sqrt{3}/2|\downarrow\rangle$ of its spin-up and spin-down states. If $|\uparrow\rangle$ and $|\downarrow\rangle$ are the eigenstates of σ_z , then what is the expectation value up to one decimal place, of the operator $10\sigma_z + 5\sigma_x$? Here, symbols have their usual meanings (A) -1.4 (B) -0.7 (C) 1.65 (D) 0.85

Ground Truth Answer: B)

Ours Longest: 2766 Tokens

NRP: 2294 Tokens

Okay, so I have this quantum mechanics problem here, and I need to find the expectation value of the operator $10\sigma_z + 5\sigma_x$ for a given spin- $1/2$ particle state. Hmm, let me try to break this down step by step. First, the state of the particle is given as a linear combination of the spin-up and spin-down states: $0.5|\uparrow\rangle + (\sqrt{3}/2)|\downarrow\rangle$. I remember that $|\uparrow\rangle$ and $|\downarrow\rangle$ are eigenstates of the Pauli matrix σ_z . So, σ_z acting on $|\uparrow\rangle$ gives $+1$ times $|\uparrow\rangle$, and σ_z acting on $|\downarrow\rangle$ gives -1 times $|\downarrow\rangle$, right? Now, the operator we're dealing with is $10\sigma_z + 5\sigma_x$. I need to find the expectation value of this operator, which is $\langle\psi|(10\sigma_z + 5\sigma_x)|\psi\rangle$. Since expectation value is linear, I can separate this into $10\langle\psi|\sigma_z|\psi\rangle + 5\langle\psi|\sigma_x|\psi\rangle$. That should make things easier. Let me handle each term separately. Starting with the first term, $10\langle\psi|\sigma_z|\psi\rangle$. I know that σ_z is diagonal in the σ_z basis, which is the same as the $|\uparrow\rangle$ and $|\downarrow\rangle$ basis. So, the expectation value of σ_z is just the weighted average of the eigenvalues, with the weights given by the square of the amplitudes in the state $|\psi\rangle$. The state $|\psi\rangle$ is $0.5|\uparrow\rangle + (\sqrt{3}/2)|\downarrow\rangle$. So, the coefficient for $|\uparrow\rangle$ is 0.5 , and for $|\downarrow\rangle$ is $\sqrt{3}/2$. The square of these gives the probabilities. Therefore, the expectation value of σ_z is $(0.5)^2(+1) + (\sqrt{3}/2)^2(-1)$. Calculating that: $(0.25)(1) + (3/4)(-1) = 0.25 - 0.75 = -0.5$. So, 10 times that would be $10(-0.5) = -5$. Okay, that's the first part. Now, the second term is $5\langle\psi|\sigma_x|\psi\rangle$. Hmm, σ_x is the Pauli X matrix, which is not diagonal in the σ_z basis. I remember that σ_x has eigenstates $|+\rangle$ and $|-\rangle$, which are superpositions of $|\uparrow\rangle$ and $|\downarrow\rangle$. Maybe I need to express $|\psi\rangle$ in the σ_x basis to compute this expectation value, or perhaps there's another way. Alternatively, I recall that σ_x can be written in matrix form as $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. So, if I can write $|\psi\rangle$ as a column vector, then σ_x acting on $|\psi\rangle$ can be computed, and then the inner product $\langle\psi|\sigma_x|\psi\rangle$ can be found. Let's represent $|\psi\rangle$ as a vector. Since $|\uparrow\rangle$ is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $|\downarrow\rangle$ is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then $|\psi\rangle$ is $0.5\begin{bmatrix} 1 \\ 0 \end{bmatrix} + (\sqrt{3}/2)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ \sqrt{3}/2 \end{bmatrix}$. So, the vector is $\begin{bmatrix} 0.5 \\ \sqrt{3}/2 \end{bmatrix}$. Now, σ_x is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. So, $\sigma_x|\psi\rangle$ is $\begin{bmatrix} 0.5\sqrt{3}/2 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.5\sqrt{3}/2 \\ 0.5 \end{bmatrix}$. Wait, that's the same as $|\psi\rangle$? Hmm, so $\sigma_x|\psi\rangle = |\psi\rangle$? [omitted] Since a and b are real in this case (the coefficients are real), this becomes $2ab$. So, for our state, $a = 0.5$, $b = \sqrt{3}/2$. Therefore, $\langle\sigma_x\rangle = 2(0.5)(\sqrt{3}/2) = \sqrt{3}/2 \approx 0.866$. Therefore, $5\langle\sigma_x\rangle = 5(\sqrt{3}/2) \approx 5(0.866) \approx 4.33$. So, adding the two terms together: $-5 + 4.33 \approx -0.67$. So, approximately -0.7 when rounded to one decimal place. Ah, okay, so that makes sense now. My initial mistake was probably in the way I computed $\sigma_x|\psi\rangle$ and then the inner product. But using the formula for expectation value in terms of coefficients, it's much simpler. So, the expectation value of σ_x is $2ab$, which is $2(0.5)(\sqrt{3}/2) = \sqrt{3}/2 \approx 0.866$. Therefore, 5 times that is approximately 4.33 , and 10 times the expectation value of σ_z is -5 . So, adding them gives approximately -0.67 , which rounds to -0.7 . So, the answer should be option B) -0.7 .

Remaining: None

NRP Ratio: 100%

ThinkPrune Longest: 8098 Tokens

NRP: 1535 Tokens

Okay, so I have this quantum mechanics problem here about a spin-half particle. Hmm, let's see. The particle is in a linear superposition given by $0.5|\uparrow\rangle + (\sqrt{3}/2)|\downarrow\rangle$. And I need to find the expectation value of the operator $10\sigma_z + 5\sigma_x$. The answer choices are A) -1.4, B) -0.7, C) 1.65, D) 0.85. Alright, first, let me recall what the expectation value of an operator is. For a quantum state $|\psi\rangle$, the expectation value $\langle O \rangle$ is given by $\langle\psi|O|\psi\rangle$. So, I need to compute $\langle\psi|(10\sigma_z + 5\sigma_x)|\psi\rangle$. Since expectation value is linear, I can separate this into $10\langle\psi|\sigma_z|\psi\rangle + 5\langle\psi|\sigma_x|\psi\rangle$. So, I need to find the expectation values of σ_z and σ_x separately and then combine them. Let me write down the given state $|\psi\rangle$. It is $0.5|\uparrow\rangle + (\sqrt{3}/2)|\downarrow\rangle$. Let me note that $|\uparrow\rangle$ and $|\downarrow\rangle$ are the eigenstates of σ_z , which is the Pauli z-matrix. So, $\sigma_z|\uparrow\rangle = +|\uparrow\rangle$ and $\sigma_z|\downarrow\rangle = -|\downarrow\rangle$. First, compute $\langle\sigma_z\rangle$. That is, $\langle\psi|\sigma_z|\psi\rangle$. Since σ_z is diagonal in the σ_z basis, which is the same as the basis in which $|\uparrow\rangle$ and $|\downarrow\rangle$ are defined, this should be straightforward. [omitted...] Looking at the options, B is -0.7 , which would be the answer. But wait, let me verify the calculations again because sometimes I might have made a mistake. First, $\langle\sigma_z\rangle$: probabilities 0.25 and 0.75 , so $0.25 - 0.75 = -0.5$. That seems right. Then, for $\langle\sigma_x\rangle$: the expectation value. Another way to compute $\langle\sigma_x\rangle$ is to note that $\sigma_x = |\rightarrow\rangle\langle\leftarrow| + |\leftarrow\rangle\langle\rightarrow|$, but maybe that's more complicated. Alternatively, for a general state $|\psi\rangle = a|\uparrow\rangle + b|\downarrow\rangle$, $\langle\sigma_x\rangle$ is $|a|^2 - |b|^2$? Wait, no, that can't be. Wait, σ_x is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, so when you compute $\langle\psi|\sigma_x|\psi\rangle$, it's $|a|^2\langle\downarrow|\sigma_x|\uparrow\rangle + |b|^2\langle\uparrow|\sigma_x|\downarrow\rangle +$ cross terms. Wait, let's compute it again step by step. $|\psi\rangle = \begin{bmatrix} 0.5 \\ \sqrt{3}/2 \end{bmatrix}$. So, in matrix form, it's a column vector $\begin{bmatrix} 0.5 \\ \sqrt{3}/2 \end{bmatrix}$. Then, $\langle\psi|\sigma_x|\psi\rangle = \begin{bmatrix} \sqrt{3}/2 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0.5 \\ \sqrt{3}/2 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & 0.5 \end{bmatrix} \begin{bmatrix} \sqrt{3}/2 \\ 0.5 \end{bmatrix} = (\sqrt{3}/2)(0.5) + (0.5)(\sqrt{3}/2) = \sqrt{3}/2 \approx 0.866$. So, that seems correct. Therefore, $5\langle\sigma_x\rangle = 5(\sqrt{3}/2) \approx 4.3301$, and $10\langle\sigma_z\rangle = -5$, so total expectation value is $-5 + 4.3301 \approx -0.6699$, which is approximately -0.67 , so -0.7 when rounded to one decimal. Therefore, the answer should be B) -0.7 .

Remaining: 5990 Tokens

[omitted] Wait, but let me think again. Is there another way to compute this? Maybe using the properties of spin states. [omitted...] So, in conclusion, $\langle\sigma_x\rangle$ is approximately 0.866 , which is $\sqrt{3}/2$. Therefore, $5\langle\sigma_x\rangle$ is $5(\sqrt{3}/2) \approx 5(0.866) \approx 4.33$. Then, $10\langle\sigma_z\rangle = -5$, as computed earlier. Therefore, the total expectation value is $-5 + 4.33 \approx -0.67$, which is approximately -0.7 when rounded to one decimal place. Therefore, the answer is B) -0.7 .

NRP Ratio: 20%