

Question

Define $p = \sum_{k=1}^{\infty} \frac{1}{k^2}$ and $q = \sum_{k=1}^{\infty} \frac{1}{k^3}$. Find a way to write $\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{(j+k)^3}$

Ours Longest: 1724 Tokens

Ground Truth Answer: $p-q$

NRP: 1191 Tokens

Okay, so I have this problem where I need to find an expression for the double sum $\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{(j+k)^3}$ in terms of $\langle p \rangle$ and $\langle q \rangle$, where $\langle p \rangle = \sum_{k=1}^{\infty} \frac{1}{k^2}$ and $\langle q \rangle = \sum_{k=1}^{\infty} \frac{1}{k^3}$. Hmm, okay, let's break this down step by step. First, I know that $\langle p \rangle$ is the well-known Basel problem, which sums to $\frac{\pi^2}{6}$, but I don't know if that's relevant here. Maybe not directly, but it's good to remember. The other sum, $\langle q \rangle$, is the Riemann zeta function at 3, so it's $\zeta(3)$, but again, not sure if that's useful yet. The main goal is to express the double sum $\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{(j+k)^3}$ in terms of $\langle p \rangle$ and $\langle q \rangle$. I need to find a clever way to relate this double sum to the single sums $\langle p \rangle$ and $\langle q \rangle$. I remember that sometimes, for sums like this, changing variables or using symmetry can help. Let me try to visualize the double sum. It's over all positive integers $\langle j \rangle$ and $\langle k \rangle$, and the term is $\frac{1}{(j+k)^3}$. So, for each pair (j, k) , we're adding up the reciprocal of the cube of their sum. I wonder if I can express this double sum as a single sum by changing the order of summation or by some substitution. Maybe if I let $n = j + k$, then for each n , I can sum over all pairs (j, k) such that $j + k = n$. That seems promising. So, let me try that. Let me fix n and consider all pairs (j, k) where $j + k = n$. For each n , $\langle j \rangle$ can range from 1 to $n - 1$, with $\langle k \rangle = n - j$. Therefore, the double sum can be rewritten as $\sum_{n=2}^{\infty} \sum_{j=1}^{n-1} \frac{1}{n^3}$. Because $j + k = n$ implies $\langle k \rangle = n - j$, and since both $\langle j \rangle$ and $\langle k \rangle$ are at least 1, n starts from 2 (since the smallest $\langle j \rangle$ and $\langle k \rangle$ can be is 1, so $1 + 1 = 2$). So, simplifying the inner sum, it's just $\sum_{j=1}^{n-1} \frac{1}{n^3}$. Since $\frac{1}{n^3}$ is constant with respect to $\langle j \rangle$, the inner sum is just $\frac{n-1}{n^3}$. Therefore, the entire double sum becomes $\sum_{n=2}^{\infty} \frac{n-1}{n^3}$. Now, I can split this into two separate sums: $\sum_{n=2}^{\infty} \frac{1}{n^2} - \sum_{n=2}^{\infty} \frac{1}{n^3}$. I know that the original definitions of $\langle p \rangle$ and $\langle q \rangle$ start from $n = 1$, so I can write these sums in terms of $\langle p \rangle$ and $\langle q \rangle$ by subtracting the $n = 1$ terms. So, $\sum_{n=2}^{\infty} \frac{1}{n^2} = p - \frac{1}{1^2} = p - 1$, and similarly, $\sum_{n=2}^{\infty} \frac{1}{n^3} = q - \frac{1}{1^3} = q - 1$. Therefore, substituting back into our expression: $(p - 1) - (q - 1) = p - 1 - q + 1 = p - q$. So, the double sum simplifies to $p - q$. I think that makes sense. So, the key steps were changing variables to $n = j + k$, then rewriting the double sum as a single sum over n , evaluating the inner sum, and then expressing the result in terms of $\langle p \rangle$ and $\langle q \rangle$. The important part was recognizing that the inner sum simplifies nicely and then using the definitions of $\langle p \rangle$ and $\langle q \rangle$ to relate it back. I don't see any mistakes in the steps, so I'm pretty confident the answer is $p - q$. Final Answer: $p - q$.

Remaining: None

LC-R1 Longest: 4287 Tokens

NRP Ratio: 100%

NRP: 3283 Tokens

Okay, so I need to find a way to express the double sum $\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{(j+k)^3}$ in terms of $\langle p \rangle$ and $\langle q \rangle$, where $\langle p \rangle = \sum_{k=1}^{\infty} \frac{1}{k^2}$ and $\langle q \rangle = \sum_{k=1}^{\infty} \frac{1}{k^3}$. Hmm, let me think about how to approach this. First, I know that $\langle p \rangle$ is the well-known Basel problem, which sums to $\frac{\pi^2}{6}$, but I don't think I need the exact value here. Similarly, $\langle q \rangle$ is the sum of reciprocals of cubes, which converges to Apéry's constant, approximately 1.202..., but again, exact value isn't necessary here. The double sum given is over all positive integers $\langle j \rangle$ and $\langle k \rangle$, and the term inside is $\frac{1}{(j+k)^3}$. So, it's a double series where each term depends on the sum of two variables. This seems a bit tricky because it's a two-dimensional sum, unlike single sums which are straightforward. Maybe I can change variables or find a way to express this double sum in terms of $\langle p \rangle$ and $\langle q \rangle$. Let me consider switching the order of summation or using some symmetry. Another idea: perhaps express $\frac{1}{(j+k)^3}$ as an integral. Since $\frac{1}{n^3}$ can be written as an integral, maybe I can use that to interchange the sum and the integral. Recall that for positive integers n , we have $\frac{1}{n^3} = \frac{1}{2} \int_0^1 \int_0^1 (xy)^{n-1} dx dy$. Wait, is that correct? Let me check. Actually, [omitted...] So, that gives us $p - q$.

Remaining: 492 Tokens

Hmm, that seems too straightforward. Wait, but is that correct? [omitted...] Which is equal to $\frac{1}{2} \left(\sum_{m=1}^{\infty} \frac{1}{m^2} - 1 \right) - \frac{1}{2} \left(\sum_{m=1}^{\infty} \frac{1}{m^3} - 1 \right) = (p - 1) - (q - 1) = p - q$. So, is that correct? Hmm. Let me verify. [omitted...] which is indeed $p - q$. Therefore, the double sum is equal to $p - q$. So, the answer is $p - q$. Final Answer: $p - q$.

NRP Ratio: 87%